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EIGENVALUE DENSITY OF CONICAL SHELLS

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SUMMARY

An analysis is made for determination of the number of eigenvalues and the eigenvalue density of conical shells over a wide range of cone geometries. The Galerkin method is used to determine a frequency equation. Values predicted by this closed form solution show favorable agreement with experimental results available in the literature. The frequency equation is then expressed in terms of wave numbers and the k-space geometry is obtained. A numerical integration procedure for determination of the cumulative number of modes from the k-space is developed. A comparison is made of values obtained from an actual count of the eigenvalues predicted by the frequency equation and this procedure shows a high degree of correlation with the k-space integration technique. Using finite differences in conjunction with the k-space integration, the eigenvalue density is obtained. The influence of variations in cone angle, thickness, and truncation are found to be significant. The results of the study are normalized with respect to geometric parameters and expressions covering a wide range of configurations are presented in graphical form.

INTRODUCTION

The problem of determining the eigenvalues of an elastic shell contained in a given frequency domain can be divided into several parts. First, the differential equations for the shell in question must be developed [1]. Secondly, the differential equations must be handled in such a way that an explicit frequency equation is obtained for the shell. In the case of the conical shell this has been done [2] using the Galerkin method in conjunction with a first order series expression for the normal displacement and the stress function which occur as the dependent variables in the differential equations. Finally using the frequency equation, the expression for the number of eigenvalues and then the modal density must be developed.

The technique used in this paper for the determination of the eigenvalues contained in a given frequency interval is the k-space integration technique [3]. The technique has been used successfully in the investigation of several other shell geometries [4,5,6].

It should be pointed out that since the frequency equation used [2] in this paper is approximate, values predicted were checked against experimental values recorded in the literature [7], and favorable agreement was obtained.

THEORETICAL DEVELOPMENT

The differential equations used to obtain a frequency equation presented in Reference [2] are as follows:

$$\frac{1}{Eh}\;\frac{m^4}{\sin^4\psi}\;\frac{\phi}{\alpha^4} - \frac{1}{\alpha\tan\psi}\;\frac{d^2w}{d\alpha^2} = 0$$

$$\frac{1}{\alpha\tan\psi}\;\frac{d^2\phi}{d\alpha^2} + D\,\frac{m^4}{\sin^4\psi}\;\frac{w}{\alpha^4} - \frac{\gamma h L^4\omega^2}{g}\;w = 0 \qquad (1)$$

where ϕ is the stress function and w is the normal displacement.

The development of these equations is based on extensional vibrations of the conical shell. It is also assumed that the mode shapes are axially symmetric and that longitudinal bending is small when compared with circumferential bending. These equations may be solved [2] by use of the Galerkin method to obtain a frequency equation of the form

$$\omega^2 = \frac{gE}{\gamma L^2} \frac{h^{*2}\left(\frac{m^4}{\sin^4 \psi} \frac{(1-\alpha_1)}{2}\right)^2 + \frac{a_n^4}{\tan^2 \psi}\left[\frac{(1-\alpha_1^4)}{8} - \frac{3(1-\alpha_1^2)}{8a_n^2}\right]^2}{\left(\frac{m^4}{\sin^4 \psi} \frac{(1-\alpha_1)}{2}\right)\left[\frac{(1-\alpha_1^5)}{10} - \frac{(1-\alpha_1^3)}{2a_n} + \frac{3(1-\alpha_1)}{4a_n^4}\right]} \tag{2}$$

Introducing the dimensionless frequency parameter λ, and the longitudinal and circumferential wave numbers k_1 and k_2 respectively, equation (2) may be written in the form

$$\lambda^2 = \frac{C_1\left(k_2^4 \frac{1-\alpha_1}{2}\right)^2 + C_2 k_1^4\left[\frac{1-\alpha_1^4}{8} - \frac{3(1-\alpha_1^2)}{8k_1^2}\right]^2}{\left(k_2^4 \frac{1-\alpha_1}{2}\right)\left[\frac{1-\alpha_1^5}{10} - \frac{1-\alpha_1^3}{2k_1^2} + \frac{3(1-\alpha_1)}{4k_1^4}\right]} \tag{3}$$

where

$$\lambda^2 = \omega^2 \frac{\gamma L^2}{gE}$$

$$C_1 = \frac{h^2}{12\ L^2\ (1-\nu^2)} \qquad C_2 = \cot^2 \psi$$

$$k_1 = a_n = \frac{n\pi}{1-\alpha_1} \qquad k_2 = \frac{m}{\sin \psi}$$

The number of eigenvalues may be expressed as the double integral, in the case of shells, over the so called k-space or wave number space defined by the frequency equation of the shell [3]. Therefore the number of eigenvalues (N) may be written in the form:

$$N \simeq \frac{1}{\Delta k_1 \ \Delta k_2} \int \int \ dk_1 \ dk_2 \tag{4}$$

Inserting the appropriate values for Δk_1 and Δk_2, rearranging terms and integrating once equation (4) becomes

$$N \frac{\pi}{\sin \psi \ (1-\alpha_1)} = \int [k_{2_u} - k_{2_1}] \ dk_1 \tag{5}$$

In the relationship given by equation (5) it should be noted that k_2 is a function of both k_1 and λ. The functional dependence involved is obtained from the frequency equation.

Rearranging terms and collecting powers of k_2 in equation (3) it is found that the frequency equation is a simple quadratic in powers of k_2^4. Therefore k_2 may be expressed explicitly as a function of k_1 and λ in the following form.

$$k_2 = \sqrt[4]{\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}} \tag{6}$$

where

$$A = C_1 \ 80 \ k_1^4$$

$$B = \lambda^2 \ [-16k_1^4 \ (1+\alpha_1+\alpha_1^2+\alpha_1^3+\alpha_1^4) + 8k_1^2 \ (1+\alpha_1+\alpha_1^2) - 120]$$

$$C = 5C_2 \ [k_1^4 \ (1+\alpha_1+\alpha_1^2+\alpha_1^3) - 3k_1^2 \ (1+\alpha_1)]^2$$

Sufficient information is now available to describe the geometry of the wave number space. In equation (6) the positive and negative square roots indicated give the upper and lower bound of the space respectively.

Figure 1 shows a typical k-space for an open or truncated cone with λ as a parameter, and Figure 2 shows a typical k-space for a closed cone with λ again as the parameter.

By equating the upper and lower bounds of the k-space given by equation (6) the limits of the space may be found as functions of the dimensionless frequency λ. The resulting expression is a cubic in k_1^2.

$$k_1^6 + Sk_1^4 + Tk_1^2 + U = 0 \qquad (7)$$

where

$$S = -\left[\frac{3(1-\alpha_1^2)}{1-\alpha_1^4} + \frac{2(1-\alpha_1^5)}{5(1-\alpha_1^4)}\ \frac{\lambda^2}{\sqrt{C_1C_2}}\right]$$

$$T = \frac{2\lambda^2}{\sqrt{C_1C_2}}\ \frac{(1-\alpha_1^3)}{(1-\alpha_1^4)}$$

$$U = -\frac{3\lambda^2}{\sqrt{C_1C_2}}\ \frac{(1-\alpha_1)}{(1-\alpha_1^4)}$$

The solution of this cubic equation in the case of the typical open cone geometry (Figure 1) yields three real roots, and the solution in the case of the typical closed cone geometry (Figure 2) yields one real root and two imaginary roots. In the latter case the real part of the imaginary roots correspond to the value of k_1 where the upper and lower bounds are at a minimum.

NUMERICAL EVALUATION

The evaluation of the integral was accomplished using numerical integration techniques with the integral in the following form.

$$N \frac{\pi}{\sin \psi \ (1-\alpha_1)} = \int_a^b \sqrt[4]{\frac{-B + \sqrt{B^2-4AC}}{2A}} \ dk_1 - \int_a^b \sqrt[4]{\frac{-B - \sqrt{B^2-4AC}}{2A}} \ dk_1 \qquad (8)$$

In equation (8) a and b represent the upper and lower bounds of the k-space as determined by the solution of equation (7), also accomplished by numerical techniques. In the case of the typical truncated cone b is the largest real root obtained, and a is the second largest. In the case of the typical closed cone, b is the one real root, and a is the real part of either imaginary root. This then corresponds to integration over the closed portion of the k-space or over the nearly closed portion.

The numerical evaluation was conducted for a variety of cone geometries so that effects of cone angle, shell thickness, and degree of truncation could be examined. Values of N corresponding to numerous values of λ were calculated so the relationship of N to λ could be plotted for each cone examined. Figures 3 through 5 show the results of these calculations.

Careful study of Figures 3 through 5 indicated that the dimensionless number of modes $\left(N \frac{\pi}{\sin \psi \ (1-\alpha_1)}\right)$ varies with changes in cone angle directly as $(\tan \psi)^{1/2}$, with changes in thickness inversely as $\frac{h}{L}$, and with changes in truncation inversely as $(1-\alpha_1)^{1/4}$. Hence the number of modes may be normalized in the following manner.

$$N \frac{\pi}{\sin \psi \ (1-\alpha_1)} \left[\frac{h(1-\alpha_1)^{1/4}}{L(\tan \psi)^{1/2}} \right] = f(\lambda) \qquad (9)$$

Figures 6 through 8 show graphically the results of this normalization procedure. Examination of the figures indicates that, with the exception of variations near the first few modes ($N \approx 1$), $f(\lambda)$ is independent of the geometry of the cone and is a function of only the dimensionless frequency.

Using finite difference techniques on the result of the k-space integration in order to obtain the density of modes with respect to frequency, a graphic representation of the modal density ($n = dN/d\lambda$) is obtained. The normalized results of these calculations are presented in Figures 9 through 11 in the following form.

$$n \frac{\pi}{\sin \psi \ (1-\alpha_1)} \left[\frac{h(1-\alpha_1)^{1/4}}{L(\tan \psi)^{1/2}} \right] = F(\lambda) \tag{10}$$

With the exception of small variations in the vicinity of values associated with $N \approx 1$, $F(\lambda)$ is also independent of geometry, as was $f(\lambda)$.

Since the k-space integration presented in this paper avoids the portion of the space near $k_1 = 0$ there is some question as to the validity of the results. In order to provide a check of the work presented here the following numerical check was used. Using equation (2), a very large number of frequencies were calculated and the number occurring below certain dimensionless frequencies was obtained and normalized. The wave numbers n and m were increased until no further increase in the count was obtained. The results of these computations were in excellent agreement with the k-space integration results. Hence the omission of a portion of the k-space in this case seems to be justifiable. It should be pointed out that the portion neglected was unbounded.

CONCLUSIONS

The number of eigenvalues and the eigenvalue density have been obtained on the bases of the frequency equation presented in this paper. Results have been normalized and presented in graphical form as independent of cone geometry. Approximate numerical expressions for the number of eigenvalues and the eigenvalue density are given by,

$$f(\lambda) = 0.876\ \lambda^{3/2} \qquad F(\lambda) = 1.31\ \lambda^{1/2}. \tag{11}$$

DEFINITION OF TERMS

E = Youngs modulus of shell material

h = Thickness of shell

α = Dimensionless coordinate = x/L

m = Number of waves in circumferential direction $m = 2, 3, 4 \ldots.$

ψ = 1/2 included cone angle

ϕ = Stress function

w = Normal displacement

D = Shell stiffness = $Eh^3/12\ (1 - \nu^2)$

γ = Density of shell

L = Length of shell (slant length to cone apex)

ω = Frequency of vibration

h^{*2} = $h^2/12\ L^2\ (1 - \nu^2) = \dfrac{D}{EhL^2}$

a_n = $n\pi/(1 - \alpha_1)$

λ = Dimensionless frequency

k_1 = Longitudinal wave number

k_2 = Circumferential wave number

N = Number of eigenvalues or modes

η = Density of eigenvalue or modal density

n = Number of longitudinal half waves $n = 1, 2, 3 \ldots$

ν = Poisions ratio

α_1 = Dimensionless truncation. L_t/L

L_t = Truncation length (slant length from top of cone to apex)

x = Coordinate along shell surface

REFERENCES

1. Vlasov, V. Z., General Theory of Shells and its Application in Engineering. NASA TT F-99, April 1964.

2. Godzevich, V. G., "Free Vibrations of Circular Conic Shells". pp. 339-343 from Theory of Plates and Shells, edited by S. M. Durgaryan, Israel Program for Scientific Translations, Jerusalem, 1966.

3. Courant, R. and Hilbert, D., Methods of Mathematical Physics, Vol. I, Interscience Publishers, Inc., New York, 1953.

4. Bolotin, V. V., "On the Density of the Distribution of Natural Frequencies of Thin Elastic Shells". PMM., Moscow 27(2): 362-364, 1963.

5. Bolotin, V. V., "The Density of Eigenvalues in Vibrational Problems of Elastic Plates and Shells". Proc. Vibr. Prob., Warsaw 6(5): 342-351, 1965.

6. Miller, D. K. and Hart, F. D., "Modal Density of Thin Circular Cylinders". NASA CR-897, December 1967.

7. Weingarten, V. I., "Free Vibrations of Conical Shells". American Society of Civil Engineers, Journal of the Engineering Mechanics Division, Vol. 91, No. 1, pp. 69-87, 1965.

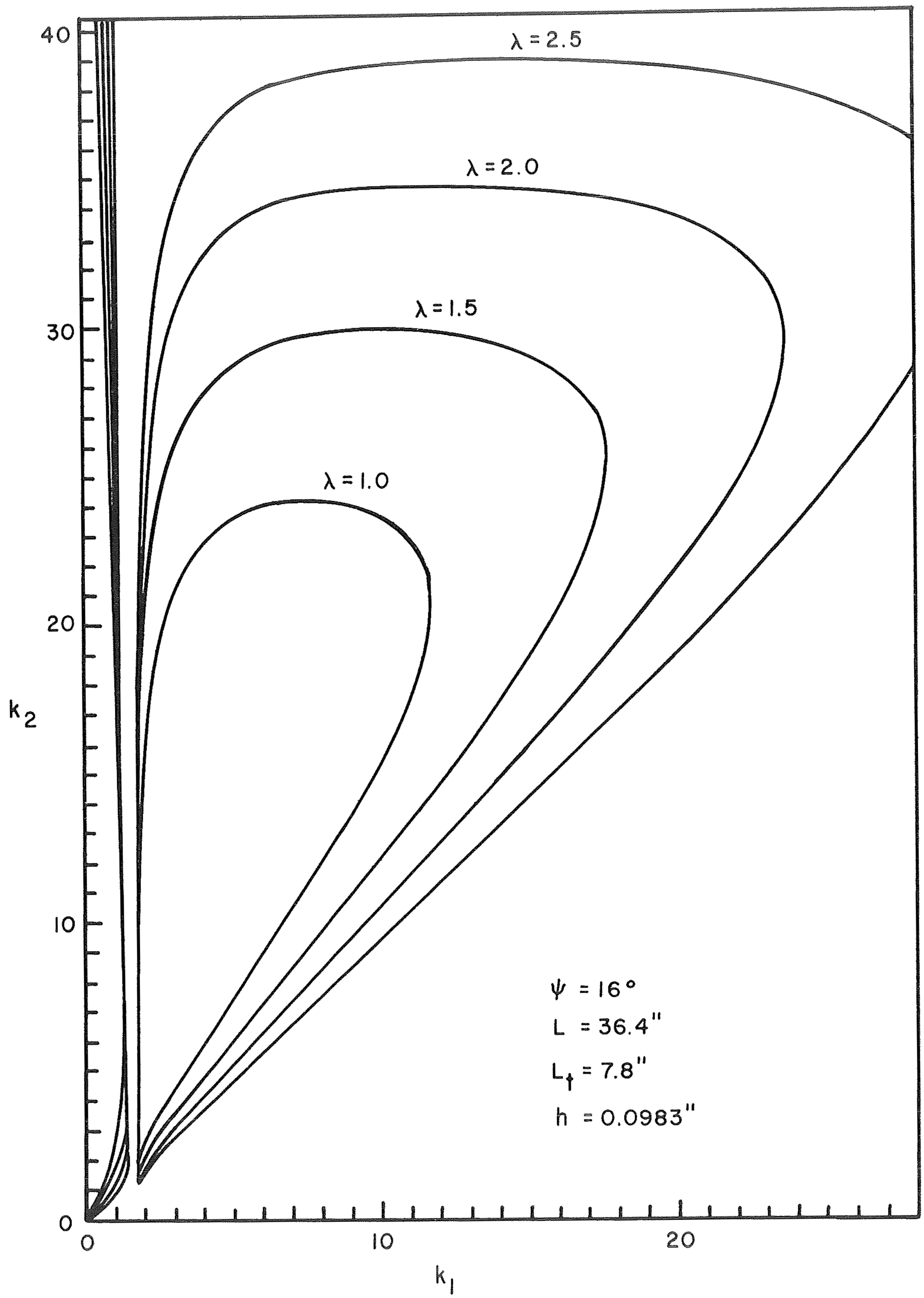

FIGURE 1. CIRCUMFERENTIAL WAVE NUMBER (k_2) VERSUS LONGITUDINAL WAVE NUMBER (k_1), PARAMETER λ

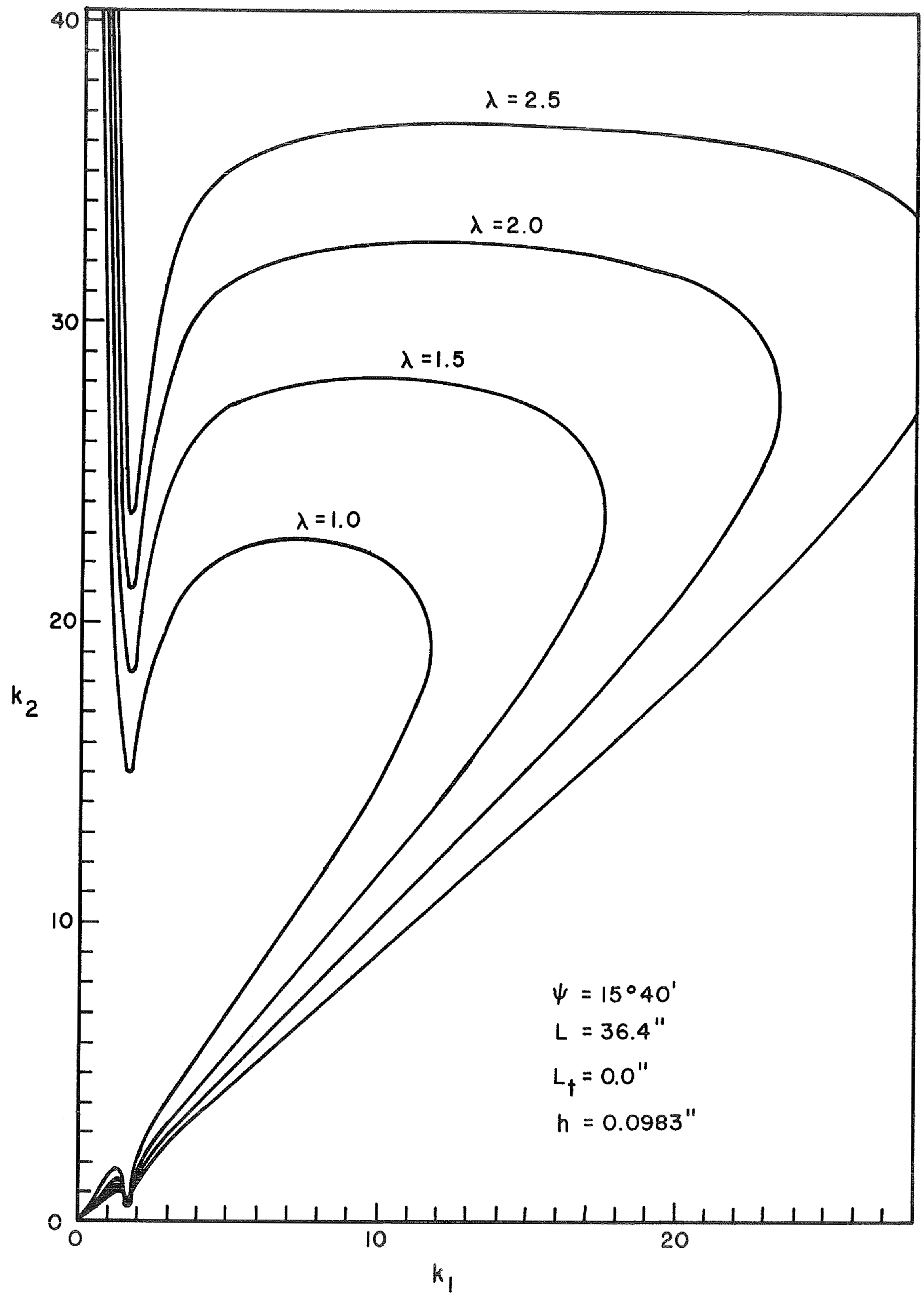

FIGURE 2. CIRCUMFERENTIAL WAVE NUMBER (k_2) VERSUS LONGITUDINAL WAVE NUMBER (k_1), PARAMETER λ

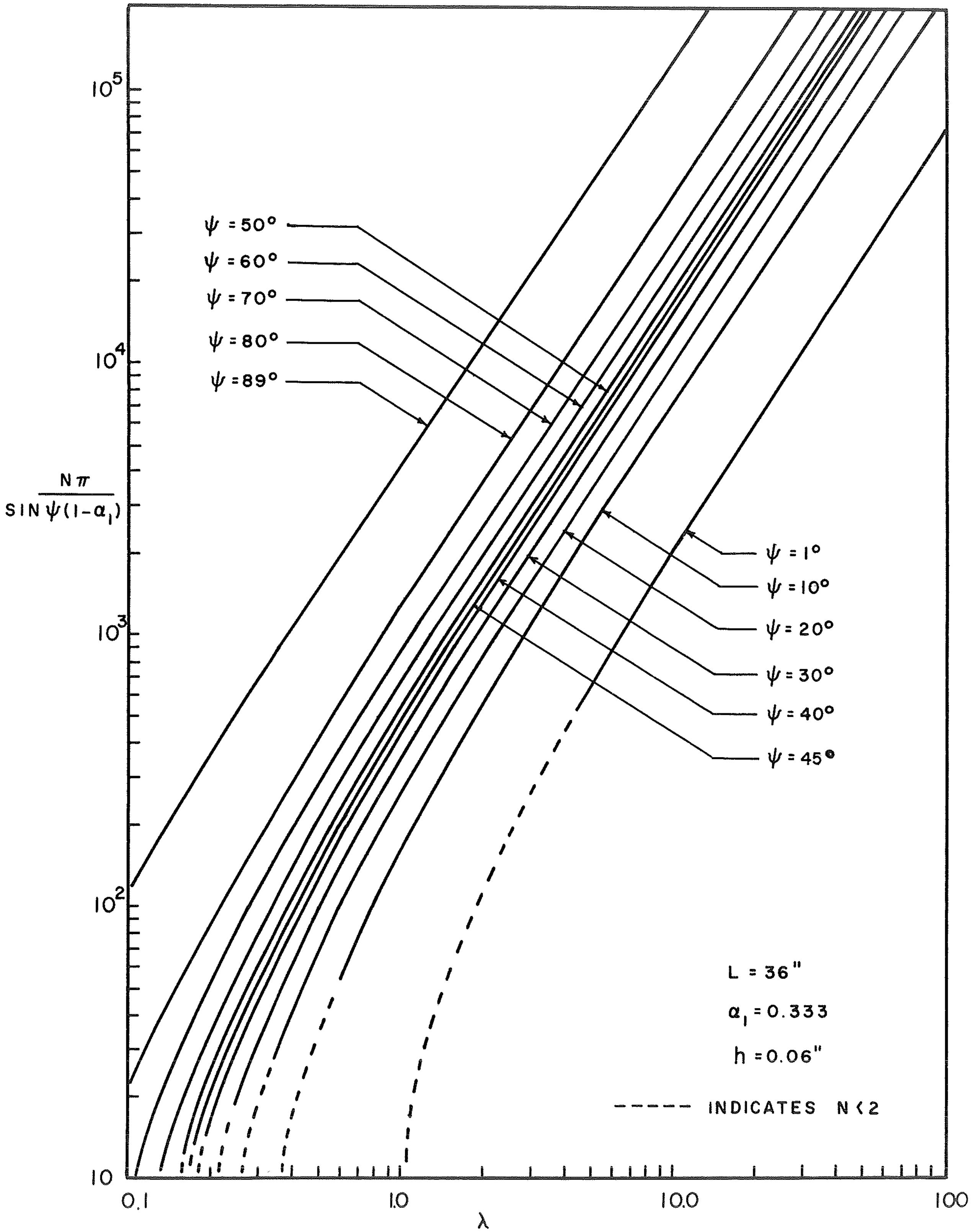

FIGURE 3. NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH ANGLE (ψ) AS THE PARAMETER

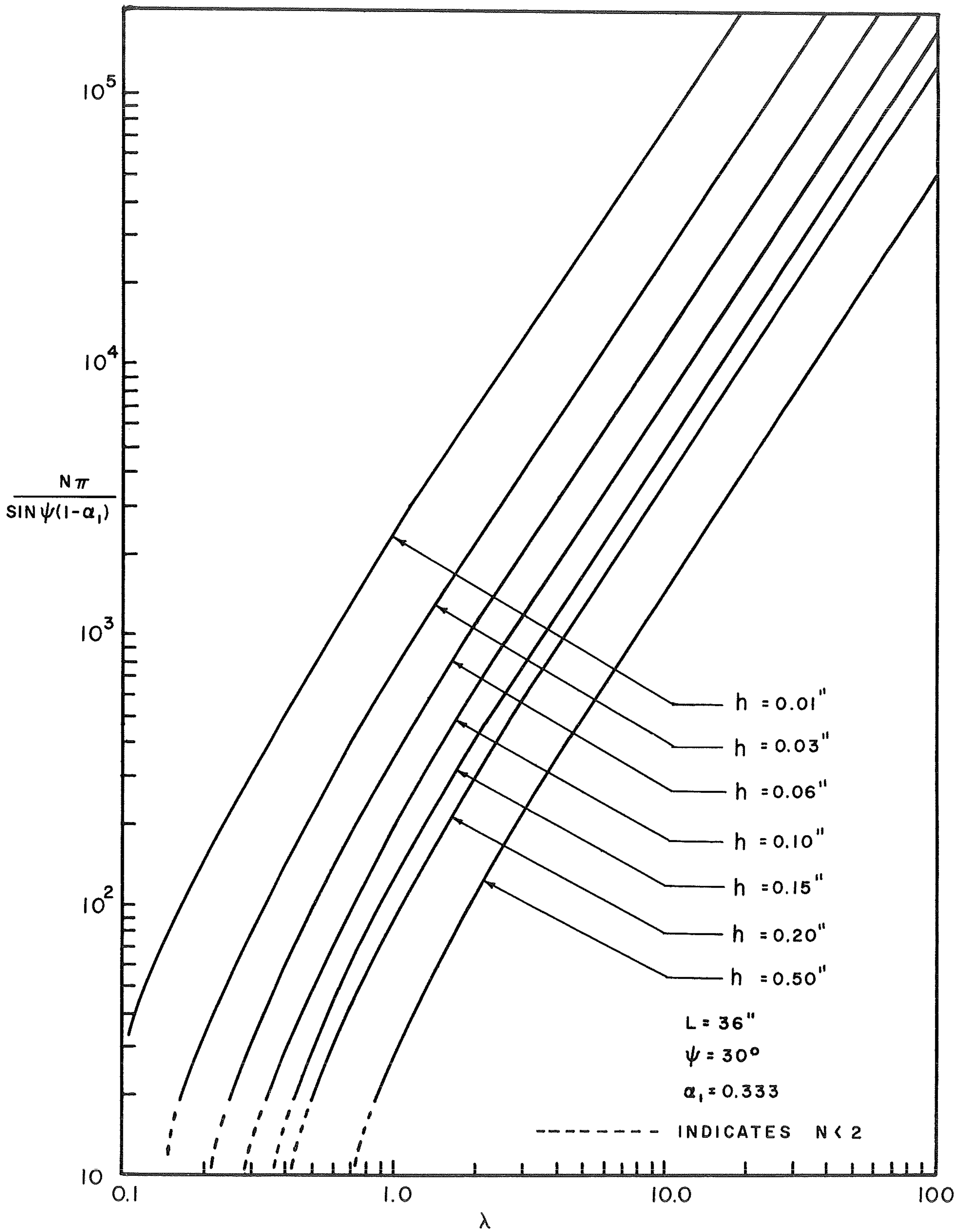

FIGURE 4. NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH THICKNESS RATIO (h/L) AS THE PARAMETER

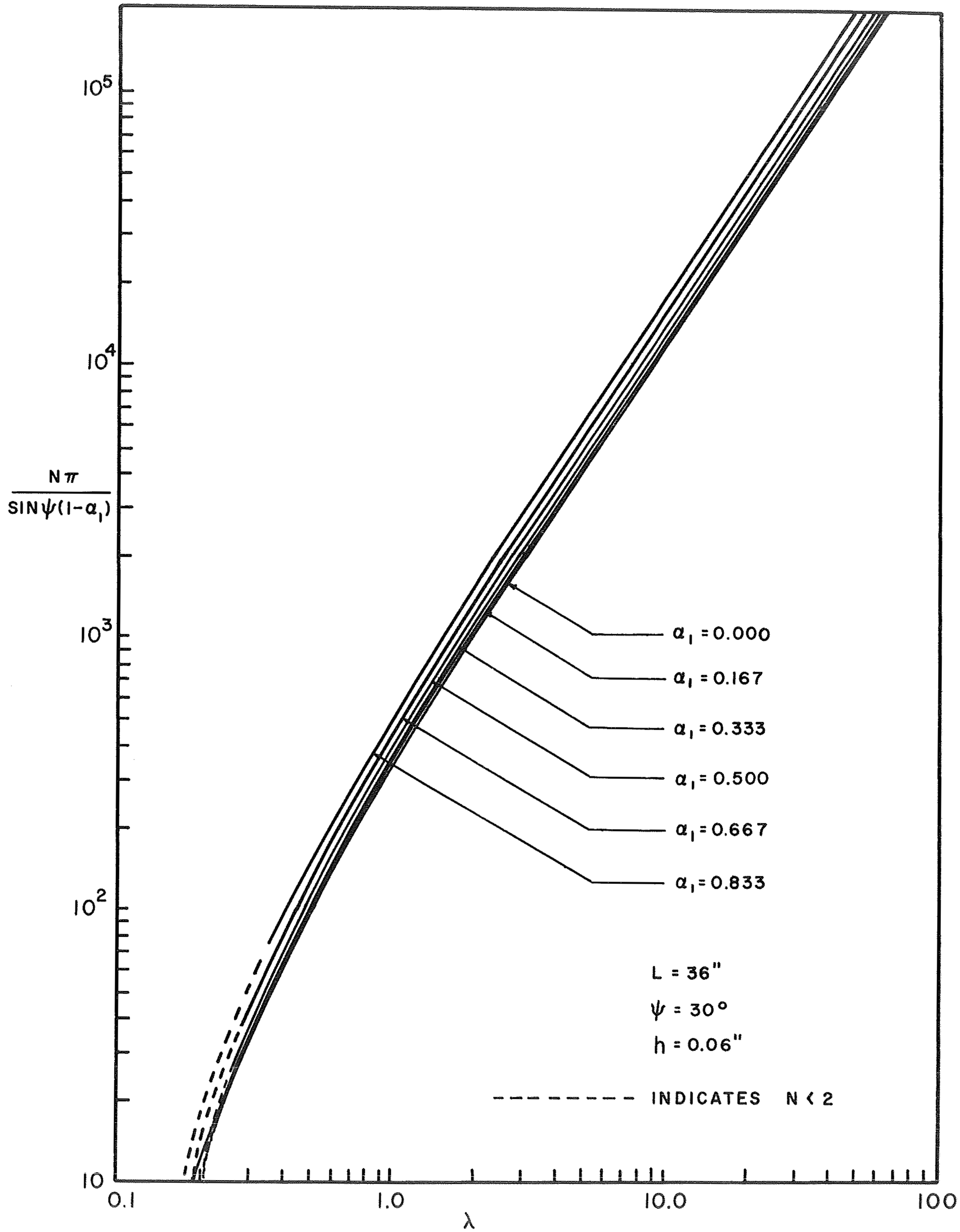

FIGURE 5. NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH TRUNCATION RATIO (α_1) AS THE PARAMETER

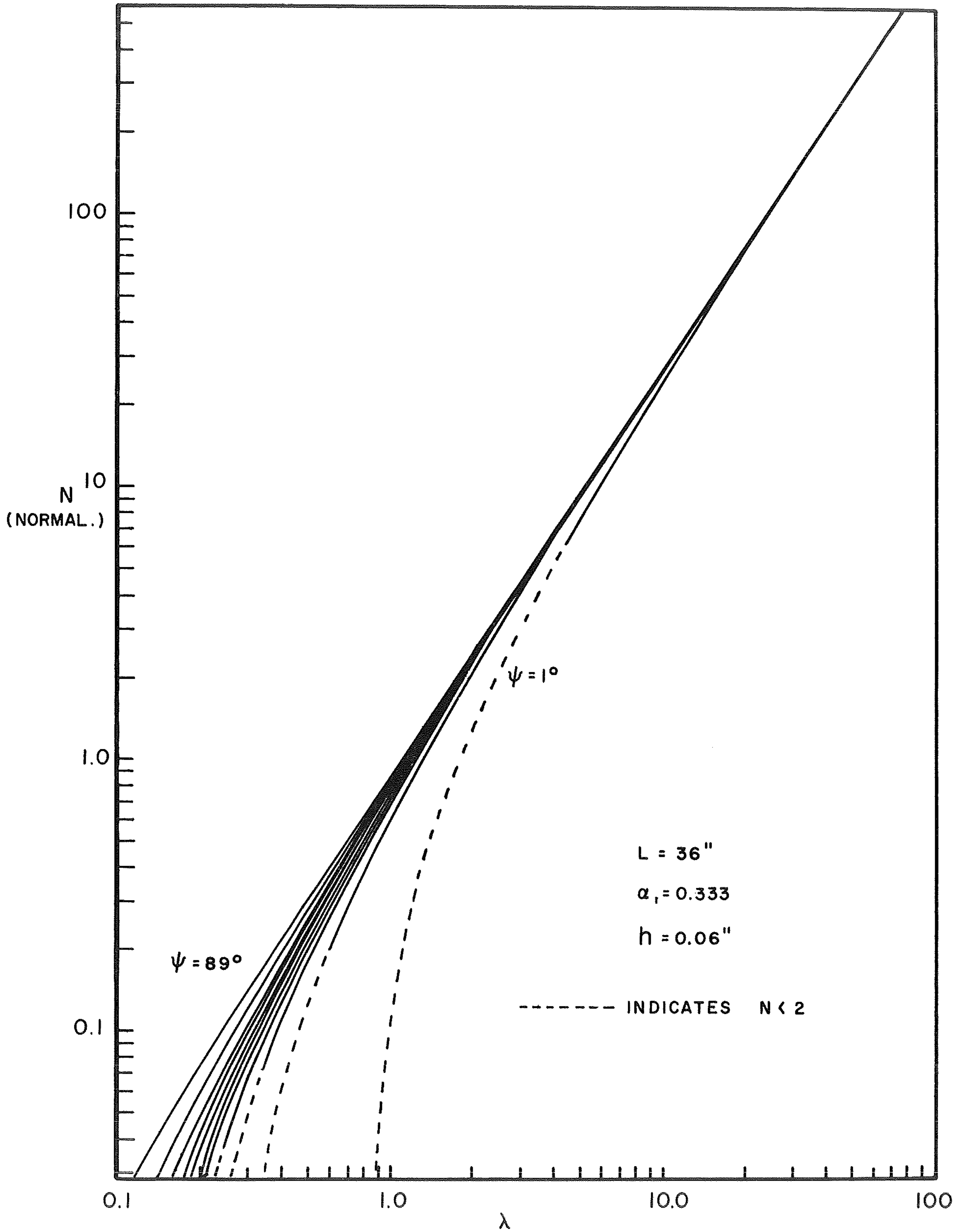

FIGURE 6. NORMALIZED NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH CONE ANGLE AS THE PARAMETER

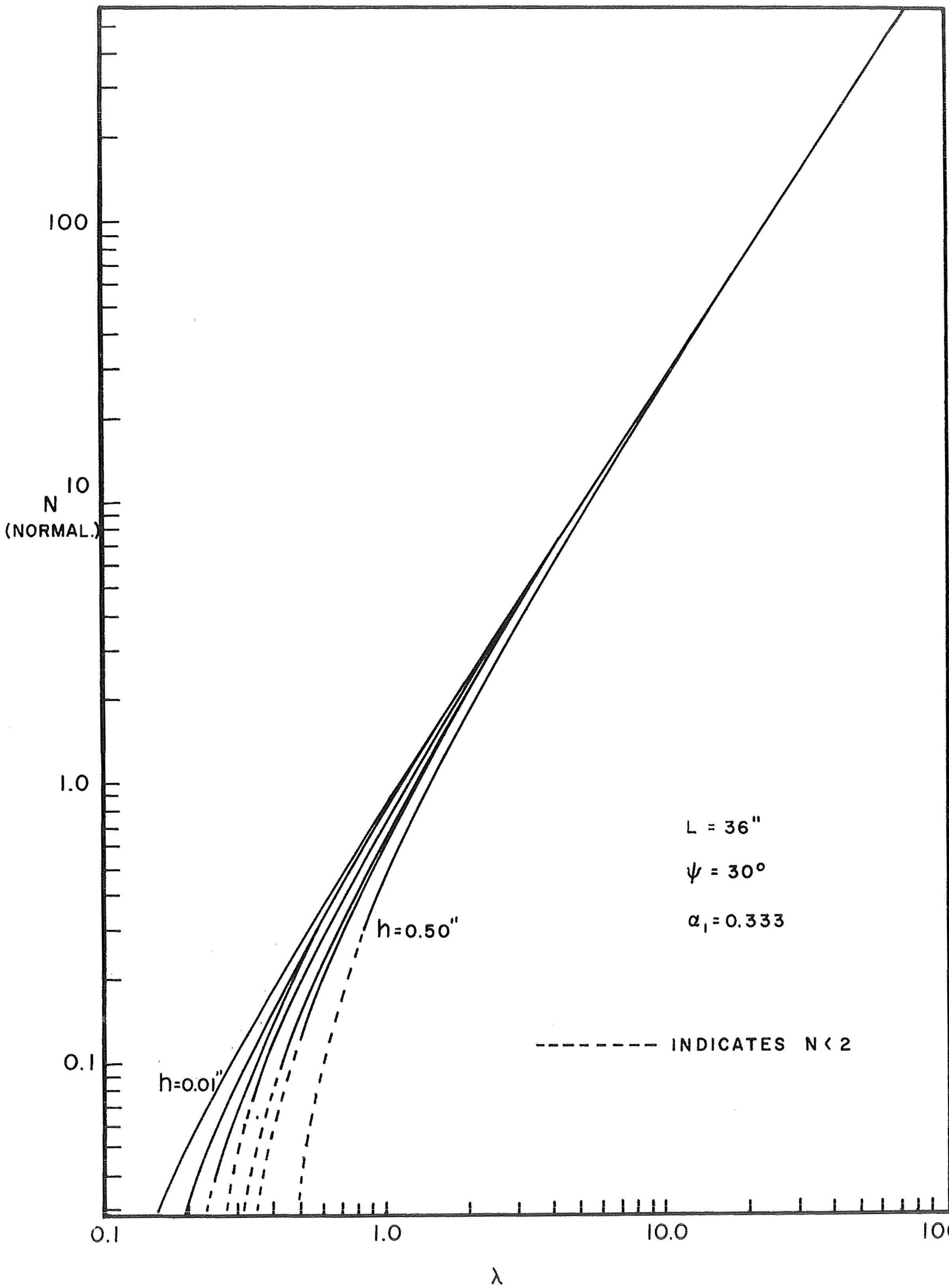

FIGURE 7. NORMALIZED NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH THICKNESS RATIO AS THE PARAMETER

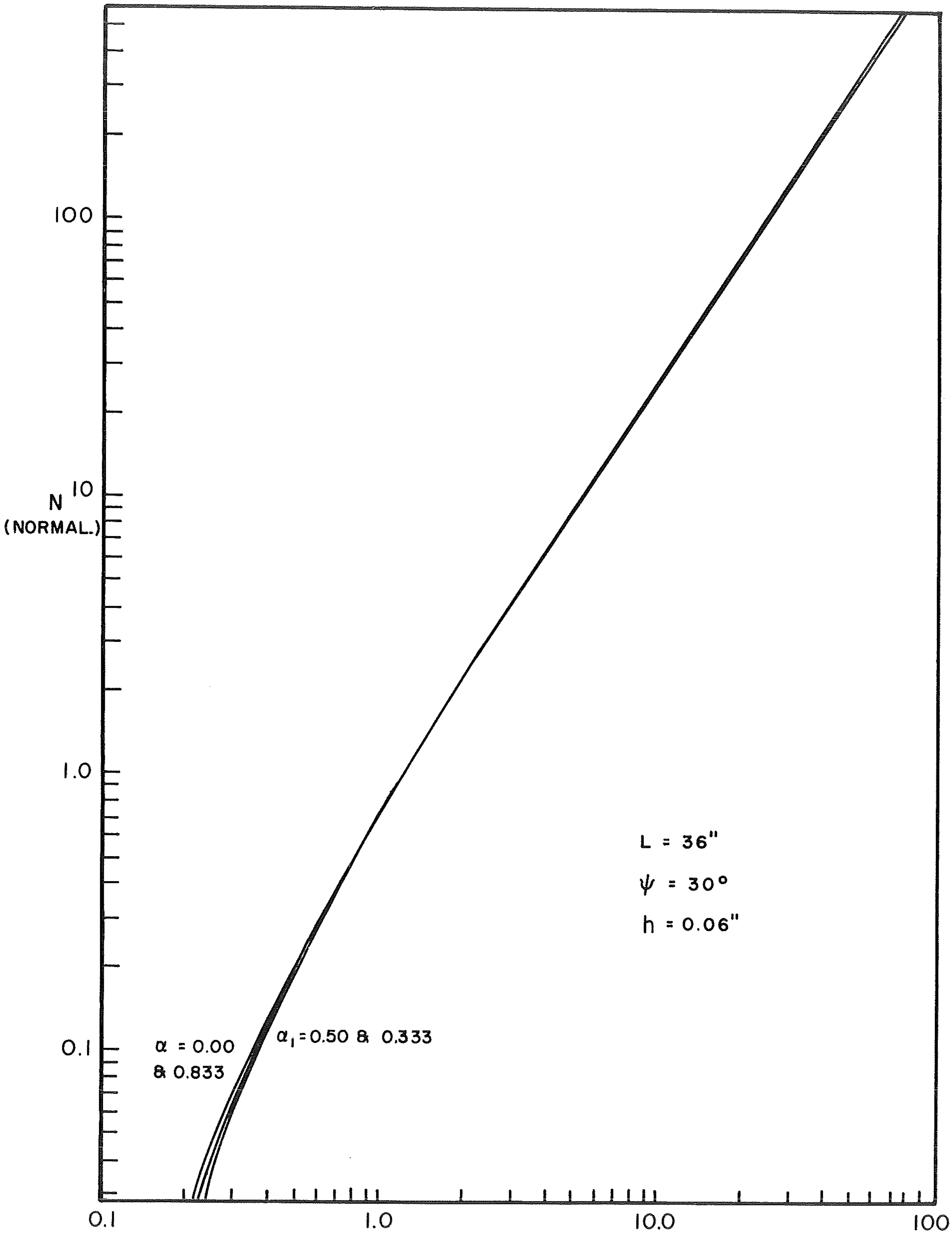

FIGURE 8. NORMALIZED NUMBER OF MODES VERSUS DIMENSIONLESS FREQUENCY WITH TRUNCATION RATIO AS THE PARAMETER

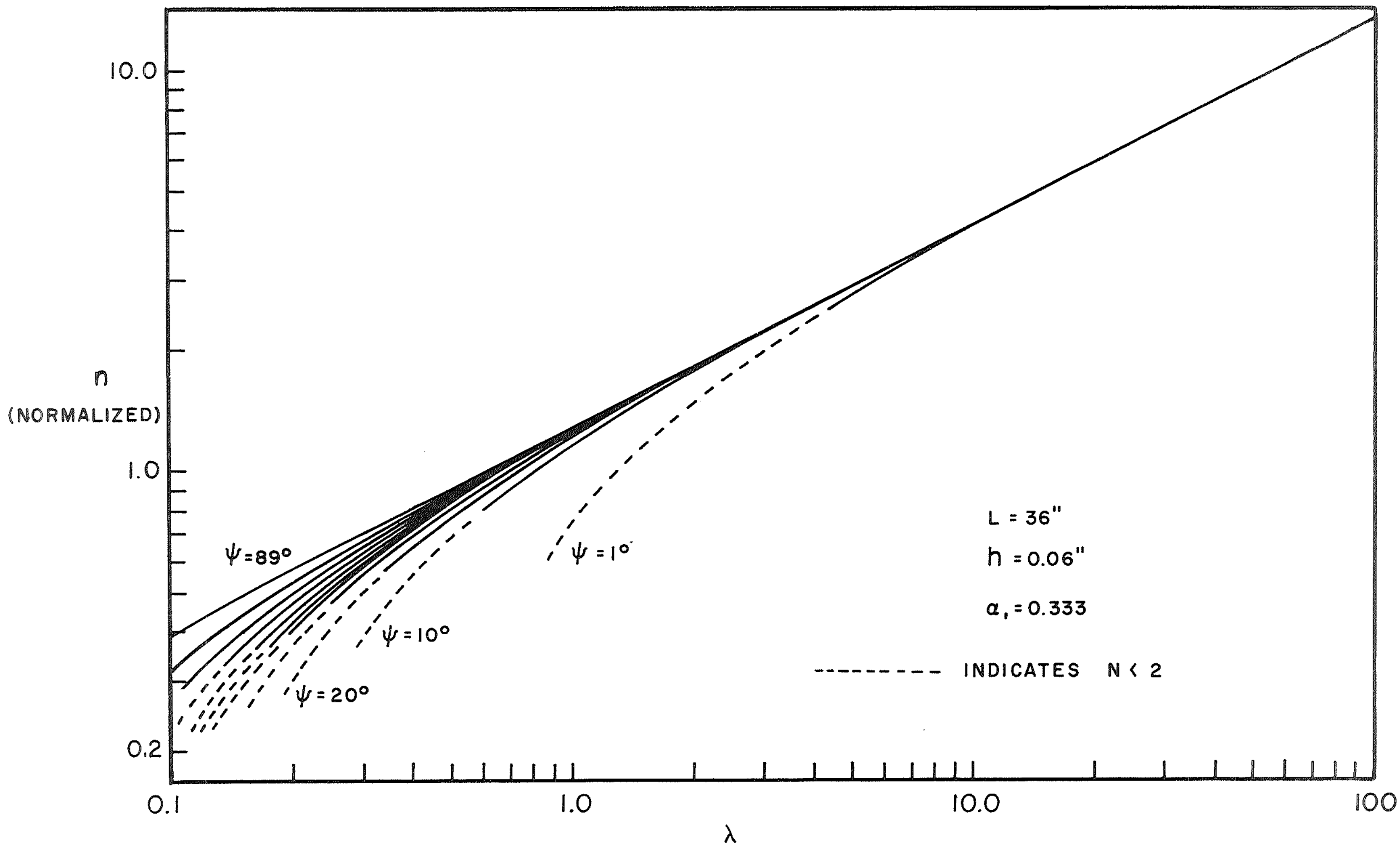

FIGURE 9. DENSITY OF MODES (NORMALIZED) VERSUS DIMENSIONLESS FREQUENCY WITH CONE ANGLE AS THE PARAMETER

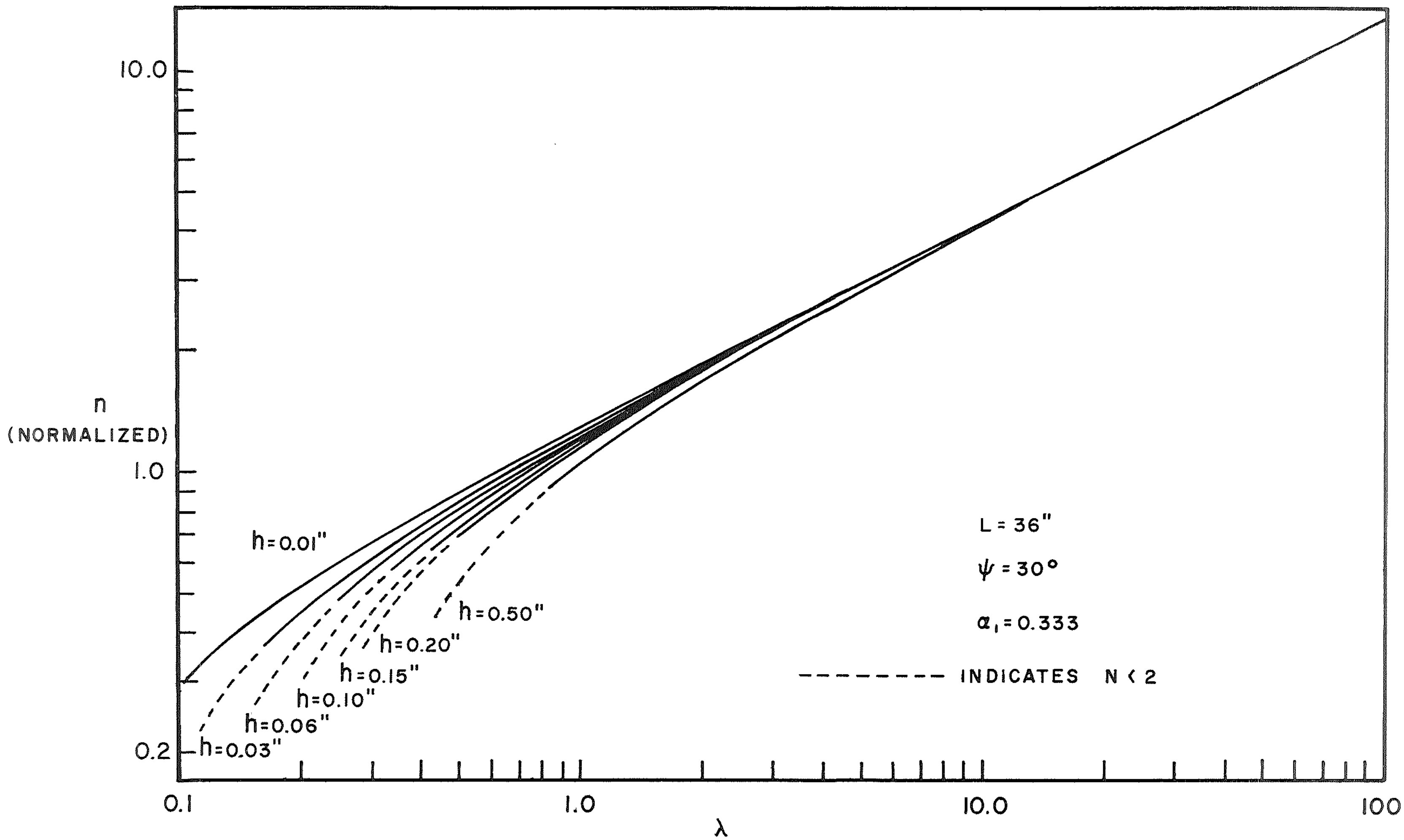

FIGURE 10. DENSITY OF MODES (NORMALIZED) VERSUS DIMENSIONLESS FREQUENCY WITH THE THICKNESS RATIO AS THE PARAMETER

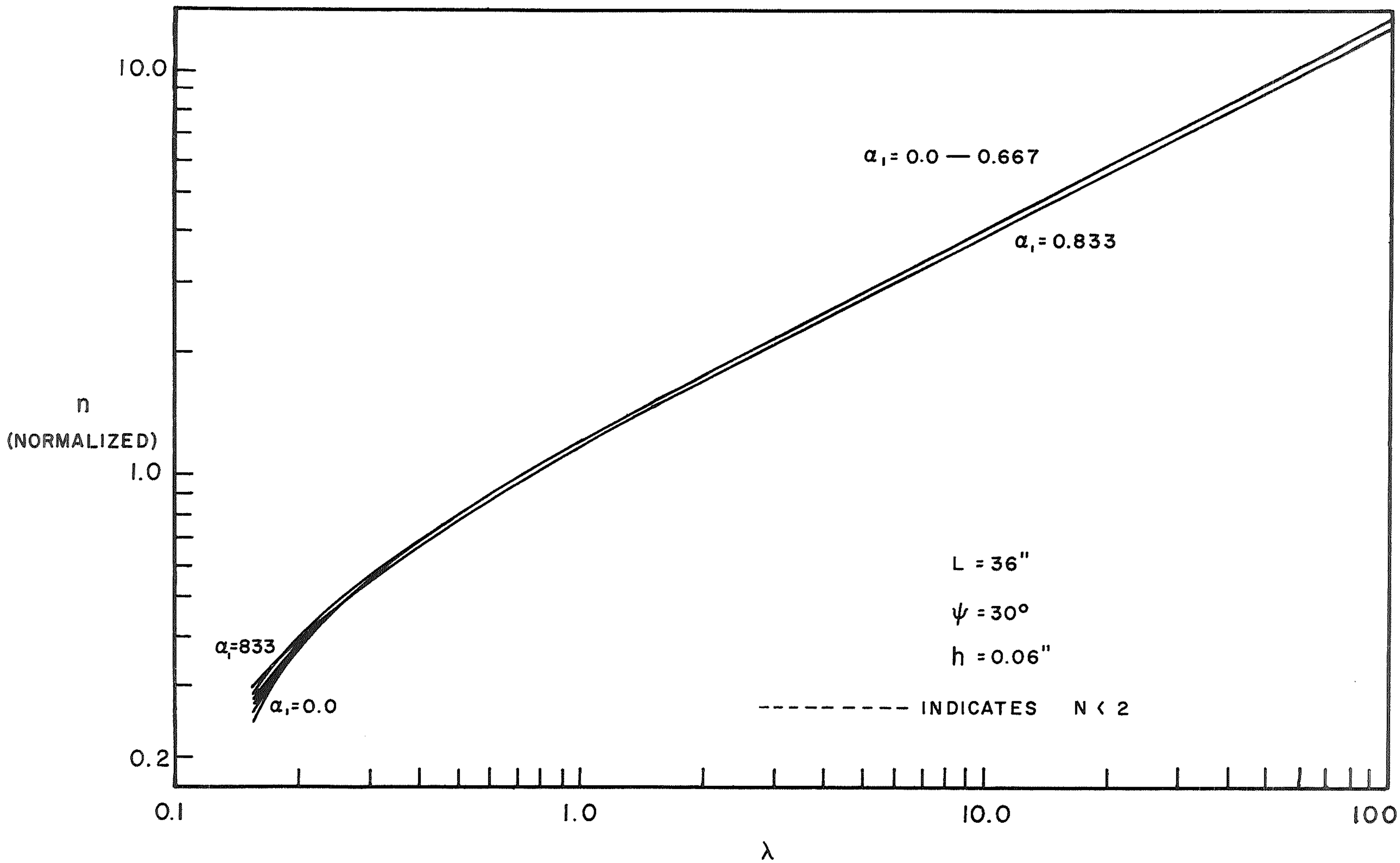

FIGURE 11. DENSITY OF MODES (NORMALIZED) VERSUS DIMENSIONLESS FREQUENCY WITH THE TRUNCATION RATIO AS THE PARAMETER